Gradient intensity selectivity for scalar image restoration using PDE

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1 Introduction

In the particular field of image restoration, non-linear or anisotropic regularization PDE’s are of primary interest. The benefit of PDE-based regularization methods lies in their ability to smooth data in a nonlinear way, allowing the preservation of important image features (contours, corners or other discontinuities). In the particular domain of scalar image restoration, a lot of studies have been presented in the literature so far: (Perona & Malik, 1990), (Alvarez et al., 1992), (Catté et al., 1992), (Geman & Reynolds, 1992), (Nitzberg & Shiota, 1992), (Cottet & Germain, 1993), (Whitaker & Pizer, 1993), (Lindeberg, 1994), (Weickert, 1995), (Deriche & Faugeras, 1996), (Krissian et al., 1997), (Weickert, 1998), (Sánchez-Ortiz et al., 1999), (Terebes et al., 2002), (Tshumperlé & Deriche, 2002), (Tschumperle & Deriche, 2005), (Histace et al., 2007), (Histace et al., 2009), (Aujol, 2009), (Elhamadi et al., 2010), to cite a few former and recent work on this particular topic.

In the particular field of medical image processing, PDE based approach for denoising are very promising tools, but generally needs to be adapted to the imaging context since noise can be of very different types (Gaussian, Poisson, Rayleigh). In (Histace et al., 2009), we showed that, considering a particular general parameterizable PDE, it was possible to integrate selectivity regarding the gradient directions to diffuse or not within the considered image. Qualitative and quantitative results were also presented on a particular medical application: enhancement of tagged cardiac MRI.

In this chapter, our aim is to extend previous work to a wider area of image processing. We propose a complementary PDE to the one presented in (Histace et al., 2009) which enables integration of selectivity regarding the intensity of the gradient to restore and which makes the preservation of thin structures from the diffusive effect possible. More precisely, we propose to make this selectivity possible thanks to the integration of a double-well potential diffusion function within the classical Perona-Malik’s PDE (Perona & Malik, 1990). That kind of approaches can be of interesting benefits for different applications in vision for robotics since for analysis of given scenes, thin structures can be of primary importance.

Our aims and motivation for such a study are mainly to show that, firstly, such a choice can lead to a stable PDE-based approach for scalar image denoising that can overpass classical approach of Perona-Malik’s from
which it is derived and which presents instability problems as formerly shown in (Catté et al., 1992). Secondly, we also want to show that this integration leads to a selective PDE-based approach that overcomes classical mean curvature or tensor driven diffusion problems, which in the particular case of directional diffusion are not suitable (see (Histace et al., 2005) and (Terebes et al., 2002)) because they tend to smooth transitions between patterns.

This chapter will be focus on PDE based approaches since this is the general framework we also want to introduce here. Given references are the main ones that will allow the reader to understand the basic of PDE based approaches. For those who are interesting in a deeper analysis of the mathematical properties od that kind of approaches, they can refer to (Aubert & Kornprobst, 2006).

This article is organized as follows: In section two, we propose some recalls about PDE-based regularization approaches. Third section is dedicated to the study of the double well function and of its mathematical properties. In fourth section, prospective results on a synthetic image and “lena” are shown and commented. Finally, the proposed approach and the obtained results are discussed in last section.

2 PDE based regularization approach: a general framework

In (Deriche & Faugeras, 1996), authors propose a global scheme for PDE-based restoration approaches. More precisely, if we denote $\psi(r, t) : \mathbb{R}^2 \times \mathbb{R}_+ \rightarrow \mathbb{R}$ the time intensity function of a corrupted image $\psi_0 = \psi(r, 0)$, the corresponding regularization problem of $\psi_0$ is equivalent to the minimization problem described by the following PDE:

$$\frac{\partial \psi}{\partial t} = c_\xi(\|\nabla \psi\|) \frac{\partial^2 \psi}{\partial \xi^2} + c_\eta(\|\nabla \psi\|) \frac{\partial^2 \psi}{\partial \eta^2},$$

(1)

where $\eta = \nabla \psi / \|\nabla \psi\|$, $\xi \perp \eta$ and $c_\xi$ and $c_\eta$ are two weighting functions (also called diffusive functions). This PDE can be interpreted as the superposition of two monodimensional heat equations, respectively oriented in the orthogonal direction of the gradient and in the tangential direction: It is characterized by an anisotropic diffusive effect in the privileged directions $\xi$ and $\eta$ allowing a non-linear denoising of scalar image.

Eq. (1) is of primary importance, for all classical methods can be expressed in that global scheme: For instance, if we consider the former anisotropic diffusive equation of Perona-Malik’s (Perona & Malik, 1990) given by

$$\frac{\partial \psi}{\partial t} = \text{div}(c(\|\nabla \psi\|)\nabla \psi),$$

(2)

with $\psi(r, 0) = \psi_0$ and $c(.)$ a monotonic decreasing function, it is possible to express it in the global scheme of Eq. (1) with

$$\left\{ \begin{array}{l}
c_\xi = c(\|\nabla \psi\|) \\
c_\eta = c'(\|\nabla \psi\|) \|\nabla \psi\| + c(\|\nabla \psi\|)
\end{array} \right.$$ 

(3)

Formulation of Eq. (1) is also interesting, for it makes stability study of classical proposed methods possible. More precisely, a stable PDE-based method for denoising will be characterized by a weighting function $c_\eta$ positive for all values of $\|\nabla \psi\|$ as formerly shown in (Catté et al., 1992). In practice, $c_\eta(.)$ function has also to be of small values in order to diffuse the data in the tangential direction of the image boundaries only.

This last property is of primary importance for the preservation of thin structures since only a “small” diffusive effect in the orthogonal direction of the corresponding gradient can lead to a low alteration of them.
What we proposed in this article is a study for the integration of a double well potential as a diffusive function $c(.)$ in Eq. (2).

3 Double well potential and corresponding PDE

3.1 Diffusive function

The double well potential considered in this article is given by the following function:

$$\phi(u) = \int_0^u v(\alpha - v)(v - 1)dv .$$

Some graphical representations of Eq. (4) for different values of $\alpha$ are proposed Fig. 1. The roots of the corresponding non linear force (i.e. $f(u) = u(\alpha - u)(u - 1)$) 0, and 1 corresponds to the local positions of the minima of the potential, whereas the root $\alpha$ represents the position of the potential maximum. The non linearity threshold $\alpha$ dethins the potential barrier between the potential minimum with the highest energy and the potential maximum.

![Figure 1: Plots of double well potential $\phi(.)$ of Eq. (4) for different values of $\alpha \in [0.5, 1]$. Solid lines stand for $\alpha = 0.5$, dash-dotted lines for $\alpha = 0.7$ and dotted lines for $\alpha = 1$.](image)

This function has to be compared with the classical Perona-Malik’s function $c_{PM}(.)$ given by:

$$c_{PM}(u) = e^{-\frac{u^2}{k^2}} ,$$

with $k$ a soft threshold defining selectivity of $c_{PM}(.)$ regarding values of image gradients. Fig. 2 shows graphical representations of $c_{PM}(.)$ dethind by Eq. (5) for different values of $k$.

As one can notice on Fig. 2.(a), for $\|\nabla\psi\| \to 0$, $c_{PM}(\|\nabla\psi\|) \to 1$, whereas for $\|\nabla\psi\| \to 1$, $c_{PM}(\|\nabla\psi\|) \to 0$. As a consequence, boundaries within images which are on a threshold, function of $k$, are preserved from the smoothing effect of Eq. (2). Regarding Fig. 1, in order to preserve this major property with integration of Eq. (4) as a diffusive function in Eq. (2), it is necessary to define this diffusive function as follows:

$$c_{DW}(u) = 1 - \phi(u) .$$
Graphical representations of $c_{DW}$ are proposed in Fig. 3 for different values of $\alpha$ parameter.

![Graphical representation of $c_{PM}(u)$](image)

**Figure 2:** Plots of function $c_{PM}(.)$ of Eq. (5) for different values of $k$. Solid lines stand for $k = 0.2$, dash-dotted lines for $k = 0.4$, and dotted lines for $k = 0.6$.

One can notice on Fig. 1 that $\phi(.)$ has been normalized. As a consequence, we are able to ensure that $0 \leq c_{DW}(u) \leq 1$ for all values of $u$ like classical PM’s function of Eq. (2). For $0 \leq \alpha < 1$, since $c_{DW}$ is issued from a double well potential, selectivity of Eq. (2) is more important and centered on a particular gradient value function of $\alpha$. For instance, for $\alpha = 0.5$, only gradients of value 0.5 are totally preserved from the diffusive effect in the tangential direction. This can be interpreted as an integration of gradient level selectivity within the restoration process.

Moreover, we are now going to show, that integration of $c_{DW}$ as diffusive function leads to interesting stability property of corresponding PDE regarding properties of the corresponding $c_\eta$ function.

### 3.2 Study of stability

It is recognized that classical Perona-Malik’s PDE presents instability problems. More precisely, as shown in (Catté et al., 1992), sometimes noise can be enhanced instead of being removed. This can be explained considering Eq. (3). If we consider $c_{PM}(.)$ function of Eq. (5), it appears that corresponding $c_{\eta pm}$ function of Eq. (3), in the global scheme of Eq. (1), can sometimes takes negative values (see Fig. 4 for illustrations). This leads to local instabilities of the Perona-Malik’s PDE which degrades the processed image instead of denoising it.

Now, if we calculate mathematical expression of $c_{\eta}$ with $c(.) = c_{DW}(.)$ of Eq. (6), one can obtain that:

$$c_{\eta dw} (\|\nabla \psi\|) = c_{DW}' (\|\nabla \psi\|) |\nabla \psi| + c_{DW} (\|\nabla \psi\|) ,$$

(7)

Taking into account that $|\nabla \psi| \in [0..1]$ and that $c_{DW}' (\|\nabla \psi\|)$ is a one-order-less polynomial function than $c_{DW} (\|\nabla \psi\|)$, it happens that:

$$c_{\eta dw} (\|\nabla \psi\|) \approx c_{DW} (\|\nabla \psi\|) = c_\xi (\|\nabla \psi\|) .$$

(8)

Considering Eq. (8), one can notice that corresponding $c_\eta$ function never takes negative values (see Fig. 3 for illustrations): Diffusive process remains stable for all gradient values of processed image which is of primary importance.
Figure 3: Plots of function $c_{DW}(\cdot)$ of Eq. (4) for different values of $\alpha$: (a) $0 < \alpha < 0.5$, (b) $0.5 < \alpha < 1$, and (c) $\alpha = 0.5$. 
Moreover, we can also notice that \( c_{\eta_{\text{DW}}} \) is exactly equal to 0 when \( c_{\xi_{\text{DW}}} = 0 \). As a consequence, thanks to a judicious choice of \( \alpha \), it becomes possible to completely stop the diffusion process in the tangential and orthogonal directions of the contours at the same time. Thin structures characterized by an identified gradient level can be preserved from any alterations.

### 4 Experimental results

We propose in this section to make a visual and quantitative comparison between classical Perona-Malik’s PDE of Eq. (2) with diffusive function \( c(. \) = \( c_{PM}(. \) of Eq. (5), and proposed derived PDE with \( c(. \) = \( c_{DW}(. \) of Eq. (6) as diffusive function. For quantitative comparisons, we will consider adapted measure of similarity between non corrupted initial image and restored one. This measure will depend on the nature of original image. For practical numerical implementations, the process of Eq. (2) is sampled with a time step \( \tau \). The restored images \( \psi(t_n) \) are calculated at discrete instant \( t_n = n\tau \) with \( n \) the number of iterations. For each experiment, the programming has been made with Matlab. The average elapsed time is between 3 or 5 seconds. If this amount of time is not compatible with a real time application (like embedded image processing algorithm) in the present version, one could easily imagine that a programming using a more dedicated language as C or C++ will make the time performances better.

#### 4.1 Synthetic image

The first proposed image is the binary image of Fig. 5.(a) corrupted by a white gaussian noise of mean zero and standard deviation \( \sigma \).

Considering binary nature of non corrupted image (Fig. 5.(a)), quantification of the denoising effect of Eq. (2) with \( c(. \) = \( c_{PM}(.) \) and \( c(. \) = \( c_{DW}(.) \), will be estimated with Fisher’s index given by
Corrupting noise is a white Gaussian one of mean zero and standard deviation $\sigma = 0.05$.

$$I_{Fisher} = \frac{(m_w - m_b)^2}{\sigma_w^2 + \sigma_b^2}, \quad (9)$$

with $m_{w,b}$ the average value of the pixels of the restored image $\psi(t_n)$ being originally in the white ($w$) or black ($b$) part of original image (Fig. 5.(a)) and $\sigma_{w,b}$ the corresponding standard deviation. Because aim of this article is to show potentiality of the described restoration method, only optimal results for both compared approaches are presented Fig. 6: Values of $k$ and $\alpha$ parameters are empirically chosen and strategy for optimal choice is not describe here.

As one can notice on Fig. 6, both visually and quantitatively, restoration of binary image of Fig. 5.(a) is better with the diffusive function of Eq. (6). More precisely, stability property of the double well function prevents restoration process from possible enhancement of corrupting Gaussian noise. Homogenous areas of Fig. 6.(b) does not visually shows oscillations, nor corners of the white square as in Fig. 6.(a). This visual impression is confirmed by variations of Fisher’s index in Fig. 6.(c) that reaches a level third times more important than with classical approach of Perona-Malik’s. The value of $\alpha$ parameter corresponding to best result is 0.5: this is not surprising, for it is also the value of the gradient intensity characterizing the boundaries of the with square. As a consequence, this experiment also confirmed the possible gradient intensity selectivity of the proposed approach interpreted as a directional diffusion process. We shall now experiment the proposed approach in the context of restoration of real scalar images.

### 4.2 Real images

We propose to compare our proposed method with PM’s approach on the classical “lena” image. For our purpose, this latter has been corrupted by a white Gaussian noise of mean zero and standard deviation $\sigma$ (see Fig. 7).

Considering nature of non corrupted image (Fig. 7.(a)), quantification of the denoising effect of Eq. (2) with $c(.) = c_{PM}(.)$ and $c(.) = c_{DW}(.)$, will be estimated with a classical PSNR measurement.

Once again, because aim of this article is to show potentiality of the described restoration method, only optimal results for both compared approaches are presented Figs. 8 and 9.
Figure 6: (a) Restored image with $c(.) = c_{PM}(.)$ (classical Perona-Malik’s approach), (b) Restored image with $c(.) = c_{DW}(.)$ (proposed approach), (c) Fisher index function of iteration number $n$, solid lines stands for classical Perona-Malik’s approach, dotted line stands for proposed method. $k$ is equal to 0.4, $\alpha$ is equal to 0.5 (these values have been empirically tuned).

One can notice on Figs. 8 and 9 that both visually and quantitatively, it is possible to find a value of $\alpha$ that can outperform results of optimal classical PM’s approach. Although the number of iterations corresponding to the optimal restoration results is more important with the proposed approach than with PM’s approach, quantitatively speaking PSNR is around 2dB higher, and visually speaking, boundaries on Fig. 8.(b) are preserved in a better way from the diffusion effect (see red circles on Fig. 8 for particular regions of interest).

5 Discussion and conclusion

In this article, we have proposed an alternative diffusive function for restoration of scalar images within the framework of PDE-based restoration approaches. The proposed diffusive function allows integrating prior knowledge on the gradient level to restore thanks parameter $\alpha$ of Eq. (6) and remains always stable on the contrary of
Figure 7: (a) Original image “lena” and (b) its corrupted version \(\psi_0\). Corrupting noise is a white Gaussian one of mean zero and standard deviation \(\sigma = 0.1\).

Figure 8: (a) Restored image with \(c(.) = c_{PM}(.)\) (classical Perona-Malik’s approach), (b) Restored image with \(c(.) = c_{DW}(.)\) (proposed approach). The red circles highlight some regions of interest where the preservation of edges are better than with Perona-Malik’s approach. \(k\) is equal to 1 for PM’s restoration approach, \(\alpha\) is equal to 0.2 for the proposed approach (these values have been empirically tuned). Time step \(\tau = 0.05\)
classical PM’s approach. Proposed method also remains fast and easy to compute. Visually and quantitatively speaking, better restoration results have been obtained, but this point must be now discussed. Since $\alpha$ parameter finally corresponds to integration of prior information about gradient level to preserve from the diffusion process, it would be interesting to make an adaptive local use of the proposed approach more than a global use.

If interesting visual and quantitative results have been obtained on “lena” image thanks to a global use of the proposed PDE, it seems that a judicious tuning of this parameter in terms of particular localization within the processed image could lead to more interesting results. Nevertheless, a strategy for a local tuning of $\alpha$ still to be now completely automatized.

Moreover, if in the proposed experiments we chose to select the gradient-level, one could also think about integrating a selectivity upon the grey-level to diffuse or not. This can be achieved by considering a variant of the proposed PDE given by

$$\frac{\partial \psi}{\partial t} = c_{DW_2} \text{div}(c_{DW_1} (||\nabla \psi||) \nabla \psi). \quad (10)$$

In this equation, $c_{DW_2}$, as shown in this article, permits a selectivity regarding gradient-level, and $c_{DW_1}$ could permit a selectivity in terms of grey-level intensity. Such an equation, integrating two diffusive constrains, could be of more interest in terms of global performance achieved by the corresponding denoising process.

Use of such double well potential function is of real interest for the PDE based image restoration approaches and can be seen as a promising topic of research (Morfu, 2009).
References


